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The R^2-action in $d = 10$ conformal supergravity

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We present the invariant action for conformal supergravity in ten dimensions. We compare our result to $d = 6$, $N = 2$ conformal supergravity, and show that in $d = 6$ a superconformal invariant based on the Gauss–Bonnet combination must exist. The contributions of the antisymmetric tensor gauge field in $d = 10$ cannot be completely expressed in terms of torsion.

1. Introduction

In this paper we derive the invariant action of $d = 10$ conformal supergravity up to terms quartic in fermions. Starting from a linearised invariant, the non-linear contributions required for full supersymmetry are obtained by the Noether method. The resulting action is unique. The gravitational degrees of freedom do not appear in the form of the Gauss–Bonnet combination.

Conformal supergravity has played an important role in the construction of matter couplings in supergravity theories. This has been the case in the development of phenomenological supergravity models, but also in the systematic study of supergravity theories in higher dimensions. The important ingredient in these applications is the fact that the large superconformal symmetry breaks up the representations of Poincaré supergravity in smaller parts, which can then be more easily put together to construct invariant actions. Gauge choices relate the two formulations of supergravity, and make it possible to go from the superconformal formulation to the physically more convenient off-shell Poincaré version.

Conformal supergravity is also interesting in itself. Conformal gravity theories may be considered as fundamental theories of gravity (see ref. [1] for a review). The use of conformal supergravity in this context requires the construction of an invariant action for the superconformal gauge multiplet, the Weyl multiplet, itself. Such actions have been known for a long time for the $N = 1$ [2] and $N = 2$ [3]

conformal supergravity theories in four dimensions. The linearised form of such an action for $N = 4$ conformal supergravity was given in ref. [3].

In dimensions > 4 a complete, non-linear conformal supergravity action was constructed only for the $d = 6$, $N = 2$ theory [4]. For $d = 10$ only an action invariant under the linearised transformations is known [5]. The problems in proceeding in $d = 10$ to a fully non-linear invariant were discussed in ref. [4]. In this paper we take up this problem again, and solve the technical difficulties involved in the construction of the $d = 10$ superconformal action.

The interest of this work is, besides the aspects mentioned above, the possible relation between the conformal supergravity actions and the low-energy limit of superstring theory.

The low-energy limit of superstring theory is given by a supergravity theory in ten dimensions. In the zero-slope limit this theory is $d = 10$, $N = 1$ Poincaré supergravity, coupled to the supersymmetric Yang–Mills multiplet [6]. The superstring induces modifications to this theory, in particular the Lorentz Chern–Simons term, which were first discovered by Green and Schwarz [7]. The purely bosonic contribution to these modifications, which are required to cancel anomalies [7], breaks the supersymmetry. However, supersymmetry can be restored by the addition of terms which depend on the fermionic fields of $d = 10$ supergravity. In the past years, much effort has been devoted to the construction of a supersymmetric version of a $d = 10$, $N = 1$ supergravity theory which includes the Lorentz Chern–Simons terms (see the extensive lists of references in ref. [8]).

The work on conformal supergravity in $d = 6$ contained an important hint on the form of this effective action. It was found in ref. [4] that in the $d = 6$, $N = 2$ conformal supergravity theory the field strength $H^{\mu\nu\rho}$ of the anti-symmetric tensor gauge field $B_{\mu\nu}$ occurs in the action as torsion, i.e. as a modification to the spin connection $\omega_\mu{}^{ab}$. This was a crucial simplifying aspect in the construction of the superconformal action. Essentially, the proper combination of ω and H transforms under supersymmetry as a Yang–Mills field, so that the construction of the effective action, known results on the Yang–Mills action itself can be used [9].

The $d = 10$, $N = 1$ Poincaré supergravity theory also contains a two-index anti-symmetric tensor gauge field, and there the use of a spin-connection with torsion turns out to be equally useful. The detailed construction of the component form of the supersymmetric low-energy effective string action in $d = 10$ [10] relied heavily on this analogy between supergravity and Yang–Mills theories.

There is a second version of $d = 10$, $N = 1$ Poincaré supergravity, which can be obtained by a duality transformation [11]. In this formulation one has the same anomaly cancellation mechanism [12]. The duality transformation replaces the two-index gauge field $B_{\mu\nu}$ by a six-index anti-symmetric tensor gauge field $A_{\mu_1 \dots \mu_6}$. The R^2-action has been constructed also in this second formulation [13]. Since the Weyl multiplet in $d = 10$ also contains such a six-index gauge field, this second formulation of supergravity appears to be closer to Poincaré supergravity. The relationship between Poincaré and conformal supergravities was explored in ref. [5] (see also ref. [14]). However, in $d = 10$ conformal supergravity one does not readily have compensating multiplets available with which one can make the transition between Poincaré and conformal supergravity. The structure of off-shell supergravity in $d = 10$ has still not been resolved beyond the linear level [15]. The relationship between the off-shell superconformal and the on-shell Poincaré R^2-action in $d = 10$ will be discussed briefly in sect. 5.

This paper is devoted to the construction of the superconformal action for the $d = 10$ Weyl multiplet. The Weyl multiplet is an off-shell multiplet containing massive spin-2 degrees of freedom. It contains $128 + 128$ bosonic and fermionic degrees of freedom, and the superconformal symmetries (dilatations, conformal boosts, S-supersymmetry, besides the usual Poincaré supersymmetry transformations) can be implemented on these fields. The field strength of the six-index gauge field of the Weyl multiplet can be represented by a three-index tensor $H_{\mu\nu\lambda}$. However, this does not imply that a torsion interpretation in conformal supergravity is possible, and indeed the simplifying methods of refs. [4,9] are not applicable. In the absence of a torsion interpretation, the construction of the action is technically complicated.

In sect. 2 of this paper we briefly review $d = 10$ conformal supergravity. It is important to note that the construction of the invariant is done in a suitable gauge, which eliminates two of the fields of the Weyl multiplet (the scalar and the spin-1/2 field). In sect. 2 we also discuss the connection between different formulations of the Weyl multiplet. Conformal supergravity in $d = 10$ is somewhat special in the sense that it is not based on a superconformal algebra, nor does it have a satisfactory superspace formulation (for a recent proposal on this last point, and further references, see ref. [16]). In this paper we consider only the component formulation presented in sect. 2. Definitions and properties of fields and curvatures that we require in this paper are gathered in appendix A.

In the linearised version of this theory one can, in an obvious way, construct two independent actions, quadratic in the fields, which are invariant under the linearised transformation rules. One of these is the purely bosonic Gauss–Bonnet combination, the other is a sum of bosonic and fermionic terms. It turns out that only one combination of these actions allows invariance under the complete non-linear transformation rules. This means that the Gauss–Bonnet combination is not separately invariant. In $d = 6$ this situation was not yet clear, since in ref. [4] no conclusions about the existence of a non-linear Gauss–Bonnet invariant were drawn.

The construction of the invariant is given in sect. 3. We give the result without the terms containing explicit gravitinos in (3.11). The complete result is presented in appendix B, eq. (B.2). In sect. 4, we discuss the relation of our result to the work in $d = 6$, $N = 2$ conformal supergravity. In particular, we show that our result implies the existence of the superconformal Gauss–Bonnet invariant in $d = 6$.

2. Conformal supergravity in $d = 10$

The Weyl multiplet is defined as the smallest *off-shell* multiplet containing the spin-2 and spin-3/2 representations of supergravity. In $d = 10$ this multiplet has 128 bosonic and 128 fermionic degrees of freedom [5,17]. These can be described in terms of the following fields: the zehnbein $e_\mu{}^a$ (45), the antisymmetric tensor gauge field $A_{\mu_1 \dots \mu_6}$ (84), and the gravitino ψ_μ (a Majorana–Weyl spinor, 144). Since these fields contain too many degrees of freedom, a bosonic and a fermionic constraint which eliminate 1 and 16 degrees of freedom, respectively, must be imposed.

The transformation rules under Q-supersymmetry read *:

$$\begin{aligned} \delta e_\mu{}^a &= \tfrac{1}{2}\bar{\epsilon}\Gamma^a\psi_\mu , \\ \delta\psi_\mu &= \mathscr{D}_\mu(\omega)\epsilon - \tfrac{1}{48}\sqrt{2}\left(\Gamma_\mu\Gamma^{(3)} + 3\Gamma^{(3)}\Gamma_\mu\right)\epsilon H_{(3)} , \\ \delta A_{\mu_1 \dots \mu_6} &= \frac{3}{4\times 6!}\sqrt{2}\,\bar{\epsilon}\Gamma_{[\mu_1 \dots \mu_5}\psi_{\mu_6]} . \end{aligned} \tag{2.1}$$

We have defined

$$H_{abc} = \tfrac{2}{3} i \epsilon^{\mu_1 \dots \mu_7}{}_{abc}\hat{R}(A)_{\mu_1 \dots \mu_7} , \tag{2.2}$$

where $\hat{R}(A)_{(7)}$ is the supercovariant curvature of $A_{(6)}$. The Bianchi identity for $R(A)$ leads to the following supercovariant restriction on H:

$$D^a H_{abc} = 0 . \tag{2.3}$$

Closure of the algebra requires the use of the constraint

$$\Gamma^{ab}\psi_{ab} = 0 , \tag{2.4}$$

where ψ_{ab} is the supercovariant gravitino curvature. The variation of (2.4) implies the bosonic condition

$$\hat{R}(\omega) - \tfrac{3}{2}H_{abc}H^{abc} = 0 . \tag{2.5}$$

In this way the superfluous bosonic and fermionic degrees of freedom are eliminated. Of course the identity (2.3) is also crucial for the off-shell closure: this multiplet definitely requires a six-index gauge field, which we prefer, where possible, to represent in the form (2.2).

In the above formulation of the Weyl multiplet the fields are invariant under scale transformations (dilatations) and S-supersymmetry transformations. These

* For spinors and γ-matrices in this paper we use the conventions of ref. [5].

local symmetries can be introduced by adding a scalar ϕ and a spin-1/2 field λ, with transformation rules

$$\delta\phi = \Lambda_{\mathrm{D}}, \qquad \delta\lambda = \eta, \tag{2.6}$$

where Λ_{D} and η are the parameters of local dilatations and S-transformations, respectively (these fields are at this stage inert under Q-transformations). Note that eqs. (2.1) and (2.6) still describe 128 + 128 degrees of freedom, and that the algebra closes off-shell. However, the commutator of two Q-transformations now contains field-dependent D- and S-transformations, which are required for closure on ϕ and λ. One may also introduce gauge fields b_μ and ϕ_μ for the D- and S-transformations, respectively, as well as K-symmetry (conformal boosts, with parameter $\Lambda_K{}^a$) and the corresponding gauge field $f_\mu{}^a$. It is convenient to choose

$$\delta b_\mu = \tfrac{1}{2}\bar{\epsilon}\phi_\mu - \tfrac{1}{2}\bar{\eta}\psi_\mu + \partial_\mu \Lambda_{\mathrm{D}} + e_\mu{}^a \Lambda_{Ka}, \tag{2.7}$$

and to turn ϕ_μ and $f_\mu{}^a$ into dependent fields by the conventional constraints

$$D_\mu \lambda = 0, \qquad D_\mu(\phi^{-1} D^a \phi) = 0. \tag{2.8}$$

The steps (2.6)–(2.8) introduce no new degrees of freedom.

If we now define a new zehnbein and gravitino by

$$e_\mu{}^a = \phi e_\mu{}^{a\prime}, \qquad \psi_\mu = \phi^{1/2}\psi'_\mu + \Gamma_a e_\mu{}^{a\prime}\lambda\phi, \tag{2.9}$$

then $e_\mu{}^{a\prime}$ has Weyl weight -1, and the new gravitino ψ'_μ Weyl weight $-\frac{1}{2}$. The gravitino also has the usual S-supersymmetry transformation $\delta\psi'_\mu = -\Gamma_\mu \eta$. Such field redefinitions, as well as a redefinition of the Q-transformations with local Lorentz and S-transformations, lead to the formulation of $d = 10$ conformal supergravity that was presented in ref. [5]. From the formulation of ref. [5] one goes back to (2.1) by the gauge choice $\phi = 1$, $\lambda = 0$.

The formulation (2.1) of conformal supergravity, with its simple transformation rules, is well suited for the purpose of constructing invariant actions, and for the coupling of conformal supergravity to matter fields (the $d = 10$, $N = 1$ Yang–Mills multiplet). Further details about covariant curvatures and their transformation rules are gathered in appendix A.

Our starting point in the construction of an invariant is the following quadratic action:

$$\begin{aligned} \mathrm{e}^{-1}\mathscr{L} = {} & a_0 R_{\mu\nu}{}^{ab}(\omega) R^{\mu\nu ab}(\omega) + a_1 R_{\mu\nu}(\omega) R^{\mu\nu}(\omega) \\ & + b_0 \bar{\psi}^{ab} \Gamma^c \mathscr{D}_c(\omega) \psi_{ab} \\ & + c_0 \mathscr{D}_\mu(\omega) H_{abc} \mathscr{D}^\mu(\omega) H^{abc}. \end{aligned} \tag{2.10}$$

Note that a possible term $a_2 R(\omega)^2$ can be eliminated by using the constraint (2.5), at the expense of a quartic term H^4. Other quadratic combinations of $\mathscr{D}H$ and R can, by partial integration, be put in the form (2.10), or can be rewritten as non-leading (cubic and quartic) terms by using eqs. (2.3) and (2.5). Similarly, other contractions of two gravitino curvatures and one derivative can be eliminated by the use of eq. (2.4) and the Bianchi identity for $\psi_{\mu\nu}$.

The requirement of invariance under global, linearised supersymmetry only determines b_0 and c_0. The reason is that the Gauss–Bonnet combination

$$\begin{aligned} \mathrm{e}^{-1}\mathscr{L}_{\mathrm{GB}} &= 6e^{\mu}{}_{[a}e^{\nu}{}_{b}e^{\lambda}{}_{c}e^{\rho}{}_{d]}R_{\mu\nu}{}^{ab}(\omega)R_{\lambda\rho}{}^{cd}(\omega) \\ &= R_{abcd}(\omega)R^{cdab}(\omega) - 4R_{ab}(\omega)R^{ba}(\omega) + R(\omega)^2, \qquad (2.11) \end{aligned}$$

transforms into a total derivative (at the linear level) for *any* variation of ω. This is due to the Bianchi identity for the Riemann tensor (A.15).

The result of the linearised calculation is

$$b_0 = \tfrac{1}{2}(a_1 + 4a_0), \qquad c_0 = -3b_0. \qquad (2.12)$$

At this stage we therefore have a two-parameter starting point for the full non-linear calculation based on the transformation rules (2.1). This will be the subject of sect. 3.

3. The construction of the action

In this section we will discuss the construction of the invariant action. The method is simple: we first make an ansatz, containing all possible terms (except quartic fermions), with arbitrary coefficients, and we then fix the coefficients by demanding invariance under the transformations (2.1).

Let us however elaborate somewhat on the starting point and on some of the intermediate stages of the calculation. The linearised result from sect. 2 is

$$\begin{aligned} \mathrm{e}^{-1}\mathscr{L} = {} & a_0 R_{\mu\nu}{}^{ab}(\omega)R^{\mu\nu ab}(\omega) + a_1 R_{\mu\nu}(\omega)R^{\mu\nu}(\omega) \\ & + \tfrac{1}{2}(a_1 + 4a_0)\bar{\psi}^{ab}\Gamma^c\mathscr{D}_c(\omega)\psi_{ab} \\ & - \tfrac{3}{2}(a_1 + 4a_0)\mathscr{D}_\mu(\omega)H_{abc}\mathscr{D}^\mu(\omega)H^{abc}. \qquad (3.1) \end{aligned}$$

Before parametrising the Noether terms, it is useful to set out the procedure to be followed. Terms in the action can be ordered in the following way. We can assign a *level* to each field, such that ω, ψ, and A each have level 1, and the zehnbein and derivatives have level 0. Then (3.1) is the action at level 2. The

TABLE 1
The schematic form of the supersymmetry transformation rules

No.	Transformation	Level change
1	$\delta\psi = \mathscr{D}\epsilon$	-1
2	$\delta\psi = H\epsilon$	0
3	$\delta H = \bar{\epsilon}\psi_{(2)}$, $\delta A = \bar{\epsilon}\psi$	0
4	$\delta\omega = \bar{\epsilon}\psi_{(2)}$	0
5	$\delta\omega = \bar{\epsilon}\psi H$	1
6	$\delta\psi_{(2)} = \epsilon R$	0
7	$\delta\psi_{(2)} = \epsilon\mathscr{D}H$	0
8	$\delta\psi_{(2)} = \epsilon HH$	1
9	$\delta e = \bar{\epsilon}\psi$	1

The symbol ψ represents the gravitino, $\psi_{(2)}$ the gravitino curvature. The fields ω, ψ and A have level 1. The complete form of these transformation rules is given in eqs. (2.1) and (A.11)–(A.14).

different terms in the supersymmetry transformations (2.1) and (A.11)–(A.14) are presented schematically in table 1. Only one contribution ($\psi_\mu \to \mathscr{D}_\mu\epsilon$) to these transformations decreases the level.

We will determine the action up to terms quartic in fermions, so that the action consists of purely bosonic terms, and of terms with two fermions. We can then ignore all variations of the action which are trilinear in fermions, since such variations would also have contributions from the unknown quartic fermion sector. In the variation of bosonic terms we may ignore all contributions that are more than linear in fermions. This means we can freely use identities for e.g. the Riemann tensor, modulo terms bilinear in fermions, but only in the *variation of the action*. In the terms in the action which are bilinear in fermions only the fermions have to be varied.

In the construction of the action we always use $\omega(e, \psi)$, i.e. the spin connection satisfies

$$\mathscr{D}_{[\mu}(\omega)e_{\nu]}{}^a = \tfrac{1}{4}\bar{\psi}_\mu\Gamma^a\psi_\nu. \tag{3.2}$$

The ψ^2 contribution to (3.2) may be safely ignored in the variation of the action. All derivatives in the action are Lorentz-covariant derivatives $\mathscr{D}_\mu(\omega(e, \psi))$. Also, we use everywhere the supercovariant tensor H_{abc} with three Lorentz indices. The Bianchi identity for H may be used with a Lorentz-covariant derivative $\mathscr{D}$, as in (A.19), in the variation of the action. Similarly, derivatives on ψ_μ are always written in terms of the supercovariant gravitino curvature ψ_{ab} (see eq. (A.22). The Riemann tensor is *not* used in supercovariant form.

It is useful to determine beforehand how partial integrations in the variation of the action will be performed. We always integrate derivatives *away* from ϵ, so that the remaining independent terms in the variation of the action never contain $\mathscr{D}_\mu\epsilon$-terms.

TABLE 2
The different structures in the variation of the action

No.	Variation	Level	Remainder
(A)	$\bar{\epsilon}\psi_{(2)}\mathscr{D}R$	2	$R \to H^2$
(B)	$\bar{\epsilon}\mathscr{D}\psi_{(2)}R$	2	$\mathscr{D}\psi_{(2)}$, $R \to H^2$
(C)	$\bar{\epsilon}\psi_{(2)}\mathscr{D}\mathscr{D}H$	2	$\mathscr{D}\mathscr{D}H \to RH$
(D)	$\bar{\epsilon}\mathscr{D}\psi_{(2)}\mathscr{D}H$	2	$\mathscr{D}\psi_{(2)}$
(E)	$\bar{\epsilon}\psi RR$	3	$R \to H^2$
(F)	$\bar{\epsilon}\psi R\mathscr{D}H$	3	$R \to H^2$
(G)	$\bar{\epsilon}\psi H\mathscr{D}R$	3	$R \to H^2$
(H)	$\bar{\epsilon}\psi_{(2)}RH$	3	$R \to H^2$, $\psi_{(2)}$ cov.
(I)	$\bar{\epsilon}\mathscr{D}\psi_{(2)}HH$	3	$\mathscr{D}\psi_{(2)}$
(J)	$\bar{\epsilon}\psi(\mathscr{D}H)^2$	3	–
(K)	$\bar{\epsilon}\psi H\mathscr{D}\mathscr{D}H$	3	$\mathscr{D}\mathscr{D}H \to RH$
(L)	$\bar{\epsilon}\psi_{(2)}H\mathscr{D}H$	3	$\psi_{(2)}$ cov.
(M)	$\bar{\epsilon}\psi RHH$	4	$R \to H^2$
(N)	$\bar{\epsilon}\psi HH\mathscr{D}H$	4	–
(O)	$\bar{\epsilon}\psi_{(2)}HHH$	4	$\psi_{(2)}$ cov.
(P)	$\bar{\epsilon}\psi HHHH$	5	–

The remainders indicate terms that may be left over after cancellation, since by the use of identities (2.5) and (A.20)–(A.22) they can be shifted to higher-level calculations. ψ indicates the gravitino, $\psi_{(2)}$ the gravitino curvature.

Our ansatz for the complete action is presented in appendix B (eq. (B.1)). The variation of (B.1) gives rise to 16 different combinations of fields and derivatives, which are presented schematically in table 2. For each of these structures we have to obtain a cancellation, but some contributions to the calculation can be moved to a higher-level variation by using constraints and identities satisfied by the fields in this theory. There are four such relations, which are indicated in the last column of table 2:

(1) $R \to HH$. The constraint (2.5).

(2) $\mathscr{D}\psi_{(2)}$. The Bianchi identity of ψ_{ab}, with a Lorentz covariant derivative. The four terms in this Bianchi identity each have a higher level than $\psi_{(2)}$, and are presented in (A.20).

(3) $\mathscr{D}\mathscr{D}H$. In some calculations we encounter the commutator of two derivatives on H. This can be written as a combination of RH-terms (eq. (A.21)).

(4) $\psi_{(2)}$ covariantization (cov.). From partial integration the combination $\mathscr{D}_{[\mu}\psi_{\nu]}$ may arise. This is rewritten, using eq. (A.22), as $\psi_{\mu\nu}$, with the appropriate covariantizations. The covariantizations contribute to a higher-level calculation.

The 16 different variations will be referred to as (A)–(P), as in table 2. Of course the Bianchi identities (A.15) for R and (A.17) for H are also used in the calculation, but these do not produce higher-level contributions.

TABLE 3
All contributions to the level-2 variations (A)–(D). The numbers in the table correspond to the supersymmetry transformations in table 1

Contribution	(A)	(B)	(C)	(D)
RR	4	–	–	–
$\bar{\psi}_{(2)}\not{\mathscr{D}}\psi_{(2)}$	–	6	–	7
$\mathscr{D}H\mathscr{D}H$	–	–	1	1
$\bar{\psi}\psi_{(2)}R$	1	1	–	–
$\bar{\psi}\psi_{(2)}\mathscr{D}H$	–	–	1	1

We will go in some detail through some of the calculations at level 3. It is at this level that we obtain a relation between the coefficients a_0 and a_1. This implies that there is only one non-linear superconformal R^2 invariant in $d = 10$.

Let us first reconsider the cancellation of the level-2 variations (A)–(D). The contributions to this calculation are shown in table 3. The entries denote the supersymmetry transformation (see table 1), which gives rise to the particular variation. Contributions come of course from all level-2 terms in the action, *and* from the level-3 terms with an explicit gravitino. Since this is the lowest level, we have no contributions from previous calculations.

The cancellation requires two relations which are contractions of the Bianchi identity (A.15) for the Riemann tensor. These are

$$\mathscr{D}_\mu(\omega)\left(eR^{ab\mu c}(\omega)\right) = 2ee^{\lambda[a}\mathscr{D}_\lambda(\omega)R^{b]c}(\omega) + \text{bilinear fermions}, \tag{3.3}$$

$$\mathscr{D}_\mu(\omega)\left(eR^{a\mu}(\omega)\right) = \tfrac{1}{2}ee^{\lambda a}\mathscr{D}_\lambda(\omega)R(\omega) + \text{bilinear fermions}. \tag{3.4}$$

In both (3.3) and (3.4) all bilinear fermions are due to (3.2). We may ignore them in the variation of the action. The terms resulting from (3.4) are moved to the next level by the use of (2.5).

In eq. (B.1) we see that there are six independent terms of the type $\bar{\psi}\psi_{(2)}R$. Their six coefficients m, n are not completely determined by this calculation. The result is

$$\begin{aligned} m_1 &= -\tfrac{1}{2}(a_1 + 4a_0), & n_1 &= 0, \\ m_2 &= \tfrac{1}{4}(a_1 + 4a_0) + \tfrac{1}{2}m_3, & n_2 &= 2m_3, \\ m_3 &= \text{free}, & n_3 &= -2m_3. \end{aligned} \tag{3.5}$$

Using eq. (3.5), the terms in the action with m and n can be rewritten in the following way:

$$\mathscr{L}_{m,n} = \tfrac{1}{4}(a_1 + 4a_0)\bar{\psi}_\mu \Gamma^{ab}\Gamma^{\mu}\psi^{cd}R_{abcd}$$
$$- \tfrac{1}{4}m_3\bar{\psi}_\mu \Gamma^{ab}\Gamma^{cdfg\mu}\psi_{fg}R_{abcd} + m_3\bar{\psi}_a\Gamma_b\psi^{ab}R. \tag{3.6}$$

The second term plays no role at level 2, since integrating away from ϵ in the variation of the gravitino gives Bianchi identities both for R and $\psi_{(2)}$. The last term is also of a higher level. We shall see that the coefficient m_3 will be determined at the next level. Note that in case $a_1 = -4a_0$ (the Gauss–Bonnet choice) the usual Noether term is absent.

The p-terms are treated in a similar fashion. The remaining terms after the cancellations (C)–(D) have the form

$$+\tfrac{1}{2}(a_1 + 4a_0)\sqrt{2}\,\mathscr{D}^d H^{abc}\left\{\tfrac{3}{2}\bar{\psi}_d\gamma_a\psi_{bc} - \tfrac{1}{6}\bar{\psi}_e\gamma_{abc}\psi_{de} + \tfrac{3}{2}\bar{\psi}_a\gamma_{bce}\psi_{de} - \tfrac{3}{2}\bar{\psi}_d\gamma_{abe}\psi_{ce}\right\}. \tag{3.7}$$

Note that in the case of the Gauss–Bonnet combination all these terms are absent.

In the p-sector in the action one can also write terms which, like the second term in eq. (3.6), do not contribute to the level-2 calculation. There are four independent terms of this type, which can be written as

$$\mathscr{D}^e H^{abc}\Big\{p_{10}\bar{\psi}^f\Gamma_{abc}\Gamma_{efgh}\psi_{gh} + p_{11}\bar{\psi}^f\Gamma_{ab}\Gamma_{efgh}\Gamma_c\psi_{gh}p_{12}\bar{\psi}^f\Gamma_a\Gamma_{efgh}\Gamma_{bc}\psi_{gh}$$
$$+p_{13}\bar{\psi}^f\Gamma_{efgh}\Gamma_{abc}\psi_{gh}\Big\}. \tag{3.8}$$

The variation of the gravitino gives after partial integration a term with $\mathscr{D}\mathscr{D}H$ which can be rewritten as RH, and a term with $\mathscr{D}\psi_{(2)}$ in the form of the Bianchi identity for the gravitino curvature. In this case the coefficients of these terms will not be determined by higher-level calculations. The reason is that in (3.8) a partial integration (with $\mathscr{D}^e$) can be performed, which expresses (3.8) in terms of other level-3 and -4 contributions. Since all such terms already appear in the ansatz, (B.1) would be overcomplete with (3.8). Therefore the four coefficients in (3.8) may be set equal to zero, and they do not appear in (B.1).

The overcompleteness of the ansatz can be recognized by the fact that free coefficients remain at the end of the calculation. A more critical aspect of the calculation is to decide which of the contributions to the *variation* of the action are independent. If a dependence between two variations is not recognized, one obtains too many equations for the input coefficients, and no solution. For this reason it is extremely useful to "integrate away from ϵ", since then the possibility

TABLE 4
All contributions to the level-3 variations (E)–(L)

Contribution	(E)	(F)	(G)	(H)	(I)	(J)	(K)	(L)
RR	9	–	5	–	–	–	–	–
$\bar{\psi}_{(2)}\not{\mathscr{D}}\psi_{(2)}$	–	–	–	–	8	–	–	–
$\mathscr{D}H\mathscr{D}H$	–	–	–	–	–	9	–	4
$\bar{\psi}\psi_{(2)}R$	6	7	–	2	–	–	–	–
$\bar{\psi}\psi_{(2)}\mathscr{D}H$	–	6	–	–	–	7	–	2
RHH	–	–	–	3	–	–	–	4
ARR	3	–	–	4	–	–	–	–
$\bar{\psi}_{(2)}\psi_{(2)}H$	–	–	–	6	–	–	–	7
$HH\mathscr{D}H$	–	–	–	–	–	–	–	3
$\bar{\psi}\psi RH$	–	1	1	1	–	–	–	–
$\bar{\psi}\psi_{(2)}HH$	–	–	–	–	1	–	–	1
$\bar{\psi}\psi H\mathscr{D}H$	–	–	–	–	–	1	1	1
(A)	–	–	–	–	–	–	–	R
(B)	B_2	B_3	–	B_1	R	–	–	–
(C)	–	–	–	$\mathscr{D}\mathscr{D}$	–	–	–	–
(D)	–	B_2	–	–	–	B_3	–	B_1

The last four lines indicate contributions which arise in the cancellation of the level-2 variations (A)–(D) (see column 4 of table 2). B_i indicate contributions from the Bianchi identity (A.20) of the gravitino curvature.

of a complicated dependence through partial integrations does not have to be considered.

This concludes the discussion of the cancellation of level-2 variations. Variations of level 3 are presented in table 4. Now we have, besides the straightforward supersymmetry transformations of terms in the action, also contributions due to remainders from the level-2 cancellations. These are indicated in the lower lines of table 4. The first stage of the level-3 calculations is to advance toward the cancellation of $\bar{\epsilon}\psi_{(2)}$ RH-terms, called (H). As we see in table 4, there are many contributions to this variation. However, the cancellations (E)–(G) already determine some of the coefficients involved, and will therefore be considered first. Particularly useful in this respect is the (E)-cancellation. This variation is relatively simple, and determines the coefficient m_3 left over from previous work, as well as the coefficient of the Lorentz Chern–Simons term in the action (all relative to a_0 and a_1). The result for this part of the action then becomes (only terms independent of H)

$$
\begin{aligned}
\mathrm{e}^{-1}\mathscr{L} = {} & a_0 R_{\mu\nu}{}^{ab}(\omega) R^{mnab}(\omega) + a_1 R_{\mu\nu}(\omega) R^{\mu\nu}(\omega) \\
& + \tfrac{1}{4}\bar{\psi}_\mu \Gamma^{ab}\left\{(a_1 + 4a_0)\Gamma^\mu \psi^{cd} + \tfrac{1}{4}a_1 \Gamma^{cdfg\mu}\psi_{fg}\right\} R_{abcd} - \tfrac{1}{4}a_1 \bar{\psi}_a \Gamma_b \psi^{ab} R \\
& + i a_0 \sqrt{2}\,\mathrm{e}^{-1}\, \epsilon^{\alpha_1 \ldots \alpha_6 \mu\nu\lambda\rho} A_{\alpha_1 \ldots \alpha_6} R_{\mu\nu}{}^{ab} R_{\lambda\rho}{}^{ab} .
\end{aligned} \tag{3.9}
$$

Note that the contribution of the Lorentz Chern–Simons term ωR (the last term in eq. (3.9) can be written as ωRH by a partial integration) is independent of a_1, and therefore occurs both in the Gauss–Bonnet combination and in the $(R_{\mu\nu ab})^2$ action. The intermediate result (3.9) was also given in ref. [4].

The (F)–(G) cancellations are sufficient to determine the coefficients t of the $\bar{\psi}\psi RH$ terms in terms of a_0 and a_1. The (G)-cancellation relates the t-coefficients in such a way that the $\bar{\epsilon}\psi H\mathscr{D}R$ variations contain $\mathscr{D}R$ in the form of a Bianchi identity. The (F)-cancellation then fixes the value of these coefficients.

The (H)-cancellation then determines the d and e coefficients, but, in particular, it fixes a_1 relative to a_0,

$$a_1 = -2a_0. \tag{3.10}$$

It is at this stage that the remaining free parameter in the action (one being an arbitrary scale is fixed.

The cancellations at level 3 ((I)–(L)), and those at levels 4 lead to unique values for all the remaining coefficients. The cancellation of the single level-5 variation is then a powerful check of the result. These and other calculations in this section have been done with the help of a computer program for algebraic manipulations. The contributions to the higher levels are presented in table 5. The complete result, with the value obtained for all coefficients, can be found in (B.2).

The contributions to the action without explicit gravitinos are ($a_0 = 1$)

$$\begin{aligned} e^{-1}\mathscr{L} = {} & R_{\mu\nu}{}^{ab}R^{\mu\nu ab} - 2R_{\mu\nu}R^{\mu\nu} + \bar{\psi}^{ab}\mathscr{D}\!\!\!/\,\psi_{ab} - 3\mathscr{D}_\mu H_{abc}\mathscr{D}^\mu H^{abc} \\ & + i\sqrt{2}\,e^{-1}\,\epsilon^{\alpha_1\ldots\alpha_6\mu\nu\lambda\rho}A_{\alpha_1\ldots\alpha_6}R_{\mu\nu}{}^{ab}R_{\lambda\rho}{}^{ab} \\ & + \sqrt{2}\,H^{abc}\Big\{6\bar{\psi}_{ab}\Gamma_d\psi_{cd} - \tfrac{3}{2}\bar{\psi}_{ad}\Gamma_b\psi_{cd} - \tfrac{1}{12}\bar{\psi}_{de}\Gamma_{abc}\psi_{de} \\ & - 3\bar{\psi}_{ad}\Gamma_{bde}\psi_{ce} - \tfrac{1}{4}\bar{\psi}_{de}\Gamma_{abcdf}\psi_{ef}\Big\} \\ & + 9R^{ab}H_{acd}H_{bcd} + 27\sqrt{2}\,H_{dae}H_{bce}\mathscr{D}^d H^{abc} \\ & + \tfrac{27}{2}H_{abe}H_{cd}{}^{e}H^{acf}H^{bd}{}_{f}. \end{aligned} \tag{3.11}$$

Some contributions to the action (compare the ansatz (B.1)), are simpler than they might have been. For instance, of the three independent combinations of H^4-terms, two have a vanishing coefficient.

Since (3.11) contains many explicit H-contributions, we should consider the possibility of rewriting the action with H-torsion. We may define

$$\Omega_\mu{}^{ab} = \omega_\mu{}^{ab} + yH_\mu{}^{ab}, \tag{3.12}$$

TABLE 5
The calculation at levels 4 ((M)–(O)) and level 5 (P). We indicate the contributions from supersymmetry variations (1–9) (see table 1), and those from previous cancellations (see table 2)

Contribution	(M)	(N)	(O)	(P)
RR	–	–	–	–
$\bar{\psi}_{(2)}\mathscr{D}\!\!\!/\,\psi_{(2)}$	–	–	–	–
$\mathscr{D}H\mathscr{D}H$	–	5	–	–
$\bar{\psi}\psi_{(2)}R$	8	–	–	–
$\bar{\psi}\psi_{(2)}\mathscr{D}H$	–	8	–	–
RHH	9	5	–	–
ARR	5	–	–	–
$\bar{\psi}_{(2)}\psi_{(2)}H$	–	–	8	–
$HH\mathscr{D}H$	–	9	4	5
$\bar{\psi}\psi RH$	2	–	–	–
$\bar{\psi}\psi_{(2)}HH$	6	7	2	8
$\bar{\psi}\psi H\mathscr{D}H$	–	2	–	–
$HHHH$	–	–	3	9
$\bar{\psi}\psi HHH$	–	1	1	2
(A)	–	–	–	–
(B)	B_4	–	–	–
(C)	–	–	–	–
(D)	–	B_4	–	–
(E)	R	–	–	R
(F)	–	R	–	–
(G)	–	R	–	–
(H)	cov.	–	R	–
(I)	B_2	B_3	B_1	B_4
(K)	$\mathscr{D}\mathscr{D}$	–	–	–
(L)	–	cov.	–	–
(M)	–	–	–	R
(O)	–	–	–	cov.

so that

$$R_{\mu\nu}{}^{ab}(\Omega) = R_{\mu\nu}{}^{ab}(\omega) + 2y\mathscr{D}_{[\mu}H_{\nu]}{}^{ab} - 2y^2H_{[\mu}{}^{ac}H_{\nu]c}{}^{b},$$

$$R_{ab}(\Omega) = R_{ab}(\omega) + y^2H_a{}^{cd}H_{bcd},$$

$$R(\Omega) = R(\omega) + y^2H_{abc}H^{abc}. \tag{3.13}$$

From the last equation in (3.13) we see that in the constraint (2.5) the H^2-term has the wrong sign to be interpreted as torsion. Also the H-contribution in the transformation rule of the gravitino cannot be absorbed into ω. Notice that the quadratic action for the Riemann tensor with torsion takes on the form

$$R_{\mu\nu}{}^{ab}(\Omega)R^{\mu\nu ab}(\Omega) = R_{\mu\nu}{}^{ab}(\omega)R^{\mu\nu ab}(\omega) + 2y^2\mathscr{D}_\mu H_{abc}\mathscr{D}^\mu H^{abc}, \quad (3.14)$$

so that in the quadratic action (3.1) H cannot be interpreted as torsion with a real value for y (we find $y^2 = -\frac{3}{2}$). Nevertheless, one can always rewrite some or all of the spin connections with torsion. However, if this cannot be done such that *all* explicit H-contributions are absorbed, it does not truly simplify the result.

A last remark about torsion concerns the Gauss–Bonnet combination (2.11). In eq. (2.11) ω can be replaced by Ω for any y, without disturbing the invariance at the linear level. The question therefore arises why in eq. (2.12) we did not find a three-parameter family of solutions, one parameter corresponding to y. The reason is that the $(\mathscr{D}H)^2$-terms arising from the Gauss–Bonnet combination with torsion cancel, so that the difference between (2.11) with ω and (2.11) with Ω consists of terms of a higher level.

4. Comparison with $d = 6$, $N = 2$ conformal supergravity

In ref. [4] an R^2-invariant was obtained for $d = 6$, $N = 2$ conformal supergravity. In this section we will discuss the implications of our result for $d = 6$. The main point we want to establish is, that the $(\mathscr{D}H)^2$ term which appears in our $d = 10$ action, when reduced to $d = 6$, can be absorbed into the (Riemann)^2 term as torsion, leaving a Gauss–Bonnet combination.

In $d = 6$, $N = 2$ conformal supergravity [18] there are two formulations of the Weyl multiplet. In one case the multiplet contains a two-index antisymmetric tensor gauge field $B_{\mu\nu}$ *, as in $d = 10$, $N = 1$ Poincaré supergravity, in the other formulation these degrees of freedom are represented by a three-index, anti-self-dual tensor. Only the first version allows the construction of a superconformal invariant [4].

The Weyl multiplet in $d = 6$, $N = 2$ conformal supergravity has 40 + 40 degrees of freedom. When the multiplet is formulated in terms of a scale-invariant sechsbein and an S-invariant gravitino, as we did in $d = 10$, these degrees of freedom are represented as follows. The bosonic fields are the sechsbein (15), the gauge field $B_{\mu\nu}$ (10), and the SU(2)-gauge field $V_\mu{}^{ij}$, symmetric in the upper indices (15). The chiral $N = 2$ gravitino has 40 degrees of freedom. The leading terms in the $d = 6$ action take the form

$$e^{-1}\mathscr{L} = R_{\mu\nu}{}^{ab}(\Omega_-)R^{\mu\nu ab}(\Omega_-) + 2\bar{\psi}^{ab}\gamma^\mu\mathscr{D}_\mu\psi_{ab} - V_{\mu\nu}{}^{ij}V^{\mu\nu ij}. \quad (4.1)$$

* In this section ten-dimensional space-time and Lorentz indices are written as $M, N, \ldots, A, B, \ldots$, six-dimensional indices will be denoted by $\mu, \nu, \ldots, a, b, \ldots$. The four-dimensional internal indices are $s, t, \ldots$.

TABLE 6
Reduction of the fields of the $d=10$ Weyl multiplet $d=6$

$d=10$	$d=6$	d.o.f.
h_{MN}	$h_{\mu\nu}$	15×1
	$h_{\mu}{}^{s}$	5×4
	h^{st}	$1\times(9+1)$
$A_{M_1\ldots M_6}$	$A_{\mu_1\ldots\mu_6}$	0
	$A_{\mu_1\ldots\mu_5}{}^{s}$	1×4
	$A_{\mu_1\ldots\mu_4}{}^{st}$	$5\times(3+3)$
	$A_{\mu_1\ldots\mu_3}{}^{stu}$	10×4
	$A_{\mu_1\mu_2}{}^{stuv}$	10×1
ψ_M	ψ_μ	80×1
	ψ^{s}	16×4

In this table the indices $M, N, \ldots$ take on the values $1,\ldots,10$, $\mu, \nu, \ldots$ the values $1,\ldots,6$ and $s, t, \ldots$ the values $1,\ldots,4$. The number of degrees of freedom (d.o.f.) in the last column are presented as a product, with the internal degrees of freedom (SO(4)) as the second factor.

In eq. (4.1) we have $\Omega_{\mu-}{}^{ab}=\omega_\mu{}^{ab}-\mathrm{H}_\mu{}^{ab}$, where H is the field strength of $B_{\mu\nu}$. $V_{\mu\nu}{}^{ij}$ is the field strength of $V_\mu{}^{ij}$. The interpretation of H as torsion can be made throughout the action. In ref. [4] a start was made with the construction of a second invariant in $d=6$, of which the leading terms form the Gauss–Bonnet combination. We will show that our result in $d=10$ implies that this second invariant exists. In principle it could be obtained from (B.2) by dimensional reduction.

In table 6 we present the $d=6$ content of the fields of the $d=10$ Weyl multiplet. The table contains $129+144$ degrees of freedom, so the constraints must still be imposed. Obviously, a truncation must be made to arrive at the $40+40$ degrees of freedom in $d=6$. In the linearised, globally supersymmetric theory we can represent the sechsbein of the $d=6$ Weyl multiplet by $h_{\mu\nu}$, a symmetric tensor gauge field. In $A_{\mu_1\ldots\mu_4}{}^{st}$ the anti-symmetric upper indices can be restricted to be (anti)-self-dual. The self-dual part is truncated, the anti-self-dual part represents the 15 degrees of freedom of $V_\mu{}^{ij}$. The field $A_{\mu\nu}{}^{stuv}$ corresponds to the 10 degrees of freedom of $B_{\mu\nu}$, the internal degrees freedom obviously giving a singlet. The gravitino is a 16-component fermion, as is the Majorana–Weyl fermion in $d=10$. In $d=6$ we can split this fermion in two 8-component chiral $d=6$ fermions, of which one chirality is truncated. When written in a Weyl basis with four components, the $N=2$ structure becomes explicit. We will not use this $N=2$ basis, so that also the relation between $A_{\mu_1\ldots\mu_4}{}^{st}$ and the SU(2) gauge field $V_\mu{}^{ij}$ will not be needed explicitly.

The $d=6$ transformation rules that follow from the transformation rules (A.1) in $d=10$ show that indeed the truncations outlined above can be done consistently. The field $B_{\mu\nu}\equiv\epsilon_{stuv}A_{\mu\nu}{}^{stuv}$ can be written as torsion in the transformation rule of the gravitino, i.e. we find $\delta\psi_\mu = \mathscr{D}(\Omega_+)\epsilon + \ldots$, with $\Omega_\pm \equiv \omega \pm T$. The torsion T is given explicitly by

$$T_{\mu\nu\lambda} = 90\sqrt{2}\,\partial_{[\mu}B_{\nu\lambda]}. \tag{4.2}$$

This requires that the $d=6$ gravitino, ψ'_μ, is defined as a suitable linear combination of ψ_μ and ψ^s of table 6. To have the usual transformation rule of the sechsbein we need to make a similar redefinition of $h_{\mu\nu}$:

$$h'_{\mu\nu} = h_{\mu\nu} + \tfrac{1}{2}\delta_{\mu\nu}h^{ss}. \tag{4.3}$$

The ten-dimensional constraints can be resolved in terms of the trace of h^{st}, and of the γ-trace of ψ^s: $\gamma^s\psi^s$. Thus the constraints no longer restrict the super-gravitational degrees of freedom. We will come back to the bosonic constraint at the end of this section.

The leading bosonic terms in the $d=10$ action (B.2) are

$$\mathscr{L} = \left(R_{MN}{}^{AB}(\omega)\right)^2 - 2\left(R_{MN}(\omega)\right)^2 - 3\mathscr{D}_M H_{ABC}\mathscr{D}^M H^{ABC}. \tag{4.4}$$

Consider the term

$$-3\mathscr{D}_M H_{ABC}\mathscr{D}^M H^{ABC} = (8\times 7!)\,\mathscr{D}_M R(A)_{N_1\ldots N_7}\mathscr{D}^M R(A)^{N_1\ldots N_7}. \tag{4.5}$$

We can work out the coefficient of the contribution of $B_{\mu\nu}$ in this expression. Expressing this in terms of the torsion (4.2) we find that the left-hand side of eq. (4.5) becomes

$$\tfrac{2}{3}\left(\mathscr{D}_a T_{bcd}\right)^2. \tag{4.6}$$

In $d=6$ the terms containing the Riemann tensor with torsion can be written as

$$\begin{aligned}\left(R_{\mu\nu}{}^{ab}(\Omega)\right)^2 &= \left(R_{\mu\nu}{}^{ab}(\omega)\right)^2 + 2\left(\mathscr{D}_a T_{bcd}\right)^2 - 2\left(\mathscr{D}_a T_{bcd}\right)\left(\mathscr{D}_b T_{acd}\right)\\ &= \left(R_{\mu\nu}{}^{ab}(\omega)\right)^2 + \tfrac{4}{3}\left(\mathscr{D}_a T_{bcd}\right)^2,\end{aligned} \tag{4.7}$$

where in the last step the Bianchi identity for the anti-symmetric tensor gauge field is used. There is still a crucial factor two between (4.6) and (4.7). This factor is correct, since we can rewrite our result (4.4), in $d=6$, in the form

$$\tfrac{1}{2}\left(R_{\mu\nu}{}^{ab}(\Omega)\right)^2 + \tfrac{1}{2}\left\{\left(R_{\mu\nu}{}^{ab}(\omega)\right)^2 - 4\left(R_{ab}(\omega)\right)^2\right\}. \tag{4.8}$$

Therefore only half of (4.7) is needed, and the remainder in (4.8) corresponds to the first two terms of the Gauss–Bonnet combination.

In (4.8) the contribution of the Riemann scalar R^2 to the Gauss–Bonnet combination is still missing. It appears when h' is used instead of h, and the bosonic constraint is resolved for h^{ss}. Let us do this analysis in some detail. The linearised form of the Riemann tensor and its contractions in $d = 10$ is given by

$$\begin{aligned} R_{MNRS}(h) &= \tfrac{1}{2}\{h_{NS,MR} - h_{MS,NR} - h_{NR,SM} + h_{MR,SN}\}, \\ R_{MN}(h) &= \tfrac{1}{2}\{h_{,MN} - h_{MR,RN} - h_{NR,RM} + \Box h_{MN}\}, \\ R(h) &= \Box h - h_{MN,MN}. \end{aligned} \tag{4.9}$$

We have used $h \equiv h^M{}_M$, and the notation $h_{MN,R} \equiv \partial_R h_{MN}$. The contributions of (4.9) to an action quadratic in R are

$$\begin{aligned} (R_{MNRS}(h))^2 &= (\Box h_{MN})^2 - 2(\Box h^{MN})(h_{MR,RN}) + (h^{MN}{}_{,MN})^2, \\ 4(R_{MN}(h))^2 &= (\Box h)^2 - 2(\Box h)(h^{MN}{}_{,MN}) + 2(\Box h_{MN})^2 \\ &\quad - 2(\Box h^{MN})(h_{MR,RN}) + 2(h^{MN}{}_{,MN})^2, \\ (R(h))^2 &= (\Box h)^2 - 2(\Box h)(h^{MN}{}_{,MN}) + (h^{MN}{}_{,MN})^2. \end{aligned} \tag{4.10}$$

When the three terms in eq. (4.10) are added in the Gauss–Bonnet combination (2.11) the result vanishes [19]. Let us now reduce the three terms in (4.10) from $d = 10$ to $d = 6$. As before we truncate $h_\mu{}^s$ and the traceless part of h^{st} (so we replace h^{st} by $\frac{1}{4}\delta^{st}h^{uu}$). Then we get the following contribution due to h^{ss} in $d = 6$:

$$\begin{aligned} (R_{MNRS}(h))^2 &\to (R_{\mu\nu\lambda\rho}(h))^2 + \tfrac{1}{4}(\Box h^{ss})^2, \\ 4(R_{MN}(h))^2 &\to 4(R_{\mu\nu}(h))^2 + 2(\Box h^{ss})R(h) + \tfrac{5}{4}(\Box h^{ss})^2, \\ (R(h))^2 &\to (R(h))^2 + 2(\Box h^{ss})R(h) + (\Box h^{ss})^2. \end{aligned} \tag{4.11}$$

The linearised form of the constraint (2.5) in $d = 10$ is $R(h) = 0$. In $d = 6$ this becomes

$$R(h) + \Box h^{ss} = 0, \tag{4.12}$$

as we can read off from the last line in (4.11). Finally we reduce our combination of (Riemann)^2 and (Ricci)^2 terms in (4.4) to $d = 6$. Using (4.11) and (4.12), we obtain

$$\begin{aligned} \mathscr{L}_{R^2} &= (R_{\mu\nu\lambda\rho}(h))^2 - 2(R_{\mu\nu}(h))^2 + \tfrac{5}{8}(\Box h^{ss})^2 \\ &= \tfrac{1}{2}(R_{\mu\nu\lambda\rho}(h))^2 + \tfrac{1}{8}(\Box h^{ss})^2 \\ &\quad + \tfrac{1}{2}\left\{(R_{\mu\nu\lambda\rho}(h))^2 - 4(R_{\mu\nu}(h))^2 + (R(h))^2\right\}, \end{aligned} \tag{4.13}$$

where we have kept the Gauss–Bonnet combination (even though it vanishes when linearised) to show how it appears in the reduction of (4.4) to $d=6$. Finally we must introduce the redefined field (4.3) for h'. This gives another contribution containing h^{ss}. We find

$$\begin{aligned}\left(R_{\mu\nu\lambda\rho}(h')\right)^2 &= \left(R_{\mu\nu\lambda\rho}(h)\right)^2 + (\Box h^{ss})R(h) + \tfrac{5}{4}(\Box h^{ss})^2 \\ &= \left(R_{\mu\nu\lambda\rho}(h)\right)^2 + \tfrac{1}{4}(\Box h^{ss})^2,\end{aligned} \tag{4.14}$$

where we again use the constraint (4.12). Finally we substitute eq. (4.14) into eq. (4.13), and find that the contribution of h^{ss} cancels. The linearised action in $d=6$ reads

$$\mathscr{L}_{R^2} = \tfrac{1}{2}\left(R_{\mu\nu\lambda\rho}(h')\right)^2 + \tfrac{1}{2}\left\{\left(R_{\mu\nu\lambda\rho}(h')\right)^2 - 4\left(R_{\mu\nu}(h')\right)^2 + \left(R(h')\right)^2\right\}. \tag{4.15}$$

The scalar h^{ss} has disappeared from the action (and from the transformation rules), and we obtain the expected form of the R^2-action. As we discussed before, (4.15) receives the required torsion contributions from the $(\mathscr{D}H)^2$-part of the action.

Thus we see that our result, reduced to $d=6$, gives the known action of ref. [4] with a torsion interpretation, *and* the Gauss–Bonnet combination. This implies that in $d=6$, $N=2$ conformal supergravity both invariants exist.

5. Discussion

The result of sect. 3 shows that the superconformal invariant in $d=10$ exists. This implies that the invariant in $d=4$, $N=4$ conformal supergravity also exists, since the two theories are related by dimensional reduction. The relation to $d=6$, $N=2$ was presented in sect. 4.

The connection between $d=10$ conformal supergravity and *off-shell* $d=10$ Poincaré supergravity is not known beyond the linear level [15]. The connection with the *on-shell* supergravity theory is known, and was discussed in detail in refs. [5,14]. If suitable restrictions are imposed on the fields of the Weyl multiplet the remaining components represent the degrees of freedom of Poincaré supergravity. A sufficient constraint to trigger this procedure is, in our notation,

$$\mathscr{D}_{[a}H_{bcd]} = 0, \tag{5.1}$$

which is the equation of motion $\mathscr{D}^{a_1}R(A)_{a_1\dots a_7} = 0$ of the six-index gauge field. By supersymmetry this implies that also

$$\mathscr{D}_{[a}\Gamma^c\psi_{b]c} = 0, \qquad \mathscr{D}_{[a}R_{b]c} = 0. \tag{5.2}$$

In ref. [5] it was found that these conditions indeed produce the on-shell Poincaré degrees of freedom.

In ref. [13] the Poincaré R^2-action with a six-index antisymmetric gauge field was obtained. Schematically, the action has the form

$$\mathscr{L} = R + \alpha \mathscr{L}_{\mathrm{P}}(R^2). \tag{5.3}$$

This action is invariant only to first order in α, and requires $\mathrm{O}(\alpha)$ modifications of the transformation rules. These arise, when in the variation of (5.3) a term proportional to an $\mathrm{O}(\alpha^0)$ equation of motion (i.e. an equation of motion arising from the R-action) is obtained. Therefore the action $\mathscr{L}_{\mathrm{P}}(R^2)$ is invariant by itself if these equations of motion are considered as constraints. In conformal supergravity we encountered a similar situation, but with the weaker constraints (2.4) and (2.5).

It was shown in ref. [5] that the transformation rules of $d=10$ conformal supergravity can be put in exactly the same form as those of $d=10$, $N=1$ Poincaré supergravity. The redefinitions which are required to put, e.g. (2.1) in the form used in ref. [13], were outlined in sect. 2. Let us assume that these redefinitions have been performed, also in the action (B.2). Then (B.2) is invariant under the same transformation rules as $\mathscr{L}_{\mathrm{P}}(R^2)$, but the invariance requires weaker constraints. Therefore, if we impose the stronger constraints corresponding to (5.1) and (5.2) on (B.2), with the redefinitions mentioned above, the result of ref. [13] will be obtained.

Is it possible to make contact between the results presented in this paper and *off-shell* Poincaré supergravity? The problem is that the transition to the off-shell Poincaré theory using the established methods requires the presence of compensating fields to break the superconformal invariance. In the absence of suitable compensating multiplets this presents interesting, but thus far unsurmountable problems.

It is a pleasure to thank E. Bergshoeff for several useful discussions.

Appendix A

CONFORMAL SUPERGRAVITY IN THE $\phi = 1$, $\lambda = 0$ GAUGE

In this appendix we gather the relevant formulae of $d = 10$ conformal supergravity. The transformation rules are

$$\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\bar{\epsilon}\Gamma^a\psi_\mu, \\
\delta\psi_\mu &= \mathscr{D}_\mu(\omega)\epsilon + \tfrac{1}{12}\sqrt{2}\left(\Gamma_\mu\Gamma^{(7)} - 3\Gamma^{(7)}\Gamma_\mu\right)\epsilon\hat{R}(A)_{(7)}, \\
\delta A_{\mu_1\ldots\mu_6} &= \frac{3}{4\times 6!}\sqrt{2}\,\bar{\epsilon}\Gamma_{[\mu_1\ldots\mu_5}\psi_{\mu_6]}.
\end{aligned} \tag{A.1}$$

Here and elsewhere we suppress explicit indices with the notation

$$\Gamma^{(7)}\hat{R}(A)_{(7)} \equiv \Gamma^{\mu_1\cdots\mu_7}\hat{R}(A)_{\mu_1\cdots\mu_7}.$$

In this paper we use the notation of ref. [5].

In transformation rules and action we prefer to work with the dual of the supercovariant curvature $\hat{R}(A)$,

$$H_{abc} = \tfrac{2}{3}i\epsilon^{\mu_1\cdots\mu_7}{}_{abc}\hat{R}(A)_{\mu_1\cdots\mu_7}. \tag{A.2}$$

The commutator of two supersymmetry transformations gives, besides the general coordinate transformation, a field-dependent Lorentz transformation,

$$\begin{aligned}[\delta(\epsilon_1),\,\delta(\epsilon_2)] &= \delta_{\rm GCT}\left(\xi^\mu = \tfrac{1}{2}\bar{\epsilon}_2\Gamma^\mu\epsilon_1\right)\\ &\quad + \delta_{\rm L}\left(\Lambda^{ab} = \tfrac{1}{2}\sqrt{2}\,\bar{\epsilon}_2\Gamma^c\epsilon_1 H^{abc} + \tfrac{1}{24}\sqrt{2}\,\bar{\epsilon}_2\Gamma^{abcde}\epsilon_1 H_{cde}\right).\end{aligned} \tag{A.3}$$

When the commutator is evaluated on the gravitino the result contains, besides the transformations (A.3), the following additional terms:

$$\begin{aligned}[\delta(\epsilon_1),\,\delta(\epsilon_2)]\psi_\mu &= (\text{A.3}) + \tfrac{37}{192}\bar{\epsilon}_2\Gamma_\mu\epsilon_1\Gamma^{ab}\psi_{ab} + \tfrac{5}{192}\bar{\epsilon}_2\Gamma^a\epsilon_1\Gamma_{a\mu}\Gamma^{cd}\psi_{cd}\\ &\quad - \tfrac{1}{1536}\bar{\epsilon}_2\Gamma_{\mu abcd}\epsilon_1\Gamma^{abcd}\Gamma^{ef}\psi_{ef}.\end{aligned} \tag{A.4}$$

Therefore we have to impose a constraint on the supercovariant gravitino curvature ψ_{ab}, which in turn implies, by supersymmetry, a constraint on the Riemann scalar,

$$\Gamma^{ab}\psi_{ab} = 0, \tag{A.5}$$

$$\hat{R}(\omega) - \tfrac{3}{2}H^{abc}H_{abc} = 0. \tag{A.6}$$

We now present a list of definitions of some of the dependent fields and curvatures and their transformation rules:

$$\begin{aligned}\omega_\mu{}^{ab} &= \omega_\mu{}^{ab}(e,\,\psi)\\ &= \omega_\mu{}^{ab}(e) - \tfrac{1}{4}\left(\bar{\psi}_\mu\Gamma^a\psi^b + \bar{\psi}^a\Gamma_\mu\psi^b - \bar{\psi}_\mu\Gamma^b\psi^a\right),\end{aligned} \tag{A.7}$$

$$R_{\mu\nu}{}^{ab}(\omega) = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_\nu{}^{cb} + \omega_\mu{}^{bc}\omega_\nu{}^{ca}, \tag{A.8}$$

$$\hat{R}(A)_{\mu_1\dots\mu_7} = \partial_{[\mu_1}A_{\mu_2\dots\mu_7]} - \frac{3}{8\times 6!}\sqrt{2}\,\bar{\psi}_{[\mu_1}\Gamma_{\mu_2\dots\mu_6}\psi_{\mu_7]}, \tag{A.9}$$

$$\psi_{\mu\nu} = \mathscr{D}_\mu\psi_\nu - \mathscr{D}_\nu\psi_\mu - \tfrac{1}{24}\sqrt{2}\left(\Gamma_{[\mu}\Gamma^{(3)} + 3\Gamma^{(3)}\Gamma_{[\mu}\right)\psi_{\nu]}H_{(3)}, \tag{A.10}$$

$$\begin{aligned}\delta\omega_\mu{}^{ab} &= \tfrac{1}{4}\bar{\epsilon}\Gamma_\mu\psi^{ab} + \tfrac{1}{2}\bar{\epsilon}\Gamma^{[a}\psi_\mu{}^{b]}\\ &\quad + \tfrac{1}{2}\sqrt{2}\,\bar{\epsilon}\Gamma_c\psi_\mu H^{abc} + \tfrac{1}{24}\sqrt{2}\,\bar{\epsilon}\Gamma^{abcde}\psi_\mu H_{cde},\end{aligned} \tag{A.11}$$

$$\delta\hat{R}(A)_{a_1\dots a_7} = -\frac{3}{8\times 6!}\sqrt{2}\,\bar{\epsilon}\Gamma_{[a_1\dots a_5}\psi_{a_6a_7]}, \tag{A.12}$$

$$\delta H_{abc} = \tfrac{1}{24}\sqrt{2}\,\bar{\epsilon}\Gamma_{abcef}\psi_{ef}, \tag{A.13}$$

$$\begin{aligned}\delta\psi_{ab} &= -\tfrac{1}{4}\Gamma_{cd}\epsilon\hat{R}_{ab}{}^{cd}(\omega)\\ &\quad + \tfrac{1}{24}\sqrt{2}\left(\Gamma_{[a}\Gamma^{(3)} + 3\Gamma^{(3)}\Gamma_{[a}\right)\epsilon D_{b]}H_{(3)}\\ &\quad + \frac{1}{12\times 48}\left(\Gamma_{[a}\Gamma^{(3)} + 3\Gamma^{(3)}\Gamma_{[a}\right)\\ &\quad \times\left(\Gamma_{b]}\Gamma^{(3)'} + 3\Gamma^{(3)'}\Gamma_{b]}\right)\epsilon H_{(3)}H_{(3)'}\end{aligned} \tag{A.14}$$

Here $\hat{R}(\omega)$ is the supercovariant version of $R(\omega)$. The derivative D is supercovariant.

The curvatures satisfy the following Bianchi identities:

$$\mathscr{D}_{[\mu}(\omega)R_{\nu\lambda]}{}^{ab}(\omega) = 0, \tag{A.15}$$

$$D_{[a_1}\hat{R}_{a_2\dots a_8]} = 0, \tag{A.16}$$

$$D^aH_{acd} = 0, \tag{A.17}$$

$$D_{[a}\psi_{bc]} = \tfrac{1}{48}\sqrt{2}\left(\Gamma_{[a}\,\Gamma^{(3)} + 3\Gamma^{(3)}\Gamma_{[a}\right)\psi_{bc]}H_{(3)}. \tag{A.18}$$

In the calculation of the superconformal invariant we use everywhere the Lorentz-covariant derivative $\mathscr{D}$. The Bianchi identity for H, eq. (A.17), is therefore used in the form

$$e^{\mu a}\mathscr{D}_\mu(\omega)H^{acd} = \text{bilinear fermions}. \tag{A.19}$$

The Bianchi identity for the gravitino curvature takes on the form

$$\begin{aligned}
\mathscr{D}_{[c}\psi_{ab]} = & \tfrac{1}{48}\sqrt{2}\left(\Gamma_{[a}\Gamma^{(3)} + 3\Gamma^{(3)}\Gamma_{[a}\right)\psi_{bc]}H_{(3)} \\
& -\tfrac{1}{4}\Gamma^{ef}\psi_{[c}R_{ab]ef} \\
& -\tfrac{1}{24}\sqrt{2}\left(\Gamma_{[a}\Gamma^{(3)} + 3\Gamma^{(3)}\Gamma_{[a}\right)\psi_b\mathscr{D}_{c]}H_{(3)} \\
& +\frac{1}{12\times 48}\left(\Gamma_{[a}\Gamma^{(3)} + 3\Gamma^{(3)}\Gamma_{[a}\right)\left(\Gamma_b\Gamma^{(3)'} + 3\Gamma^{(3)'}\Gamma_b\right)\psi_{c]}H_{(3)}H_{(3)'} \\
& + \text{trilinear fermions}. \qquad \text{(A.20)}
\end{aligned}$$

The four contributions to (A.20) are referred to as B_1–B_4, respectively, in tables 4 and 5.

The constraint (A.6), the identity (A.20), the relation

$$\left[\mathscr{D}_\mu(\omega), \mathscr{D}_\nu(\omega)\right]H^{abc} = -3R_{\mu\nu}{}^{[a}{}_d(\omega)H^{bc]d}, \qquad \text{(A.21)}$$

and the supercovariantizations in

$$\mathscr{D}_\mu\psi_\nu - \mathscr{D}_\nu\psi_\mu = \psi_{\mu\nu} + \tfrac{1}{24}\sqrt{2}\left(\Gamma_{[\mu}\Gamma^{(3)} + 3\Gamma^{(3)}\Gamma_{[\mu}\right)\psi_{\nu]}H_{(3)}, \qquad \text{(A.22)}$$

link the different steps in the determination of the invariant action. This is explained in detail in sect. 3.

Appendix B

THE COMPLETE RESULT

In this appendix we present the complete result for the $d = 10$ superconformal action. The action is first written in the form of an ansatz, in which arbitrary coefficients appear. These have been determined with the procedure explained in sect. 3. Finally we write the action again, this time substituting the calculated values for the coefficients.

The ansatz for the $N = 1$, $d = 10$ conformal supergravity action, in the gauge $\lambda = 0$, $\phi = 1$, reads

$$\begin{aligned}
e^{-1}\mathscr{L} = & a_0 R_{\mu\nu}{}^{ab}R^{\mu\nu ab} + a_1 R_{\mu\nu}R^{\mu\nu} + b_0\bar{\psi}^{ab}\not{\mathscr{D}}\psi_{ab} + c_0\mathscr{D}_\mu H_{abc}\mathscr{D}^\mu H^{abc} \\
& + R^{abcd}\left\{m_1\bar{\psi}_a\Gamma_b\psi_{cd} + m_2\bar{\psi}_e\Gamma_{abe}\psi_{cd} + m_3\bar{\psi}_a\Gamma_{cde}\psi_{be}\right\}
\end{aligned}$$

$$+R^{ab}\Big\{n_1\bar{\psi}_c\Gamma_a\psi_{bc}+n_2\bar{\psi}_a\Gamma_c\psi_{bc}+n_3\bar{\psi}_c\Gamma_{cad}\psi_{bd}\Big\}$$

$$+\mathscr{D}^dH^{abc}\Big\{p_1\bar{\psi}_d\Gamma_a\psi_{bc}+p_2\bar{\psi}_a\Gamma_b\psi_{dc}$$

$$+p_3\bar{\psi}_e\Gamma_{dae}\psi_{bc}+p_4\bar{\psi}_e\Gamma_{abe}\psi_{dc}+p_5\bar{\psi}_e\Gamma_{dab}\psi_{ce}+p_6\bar{\psi}_e\Gamma_{abc}\psi_{de}$$

$$+p_7\bar{\psi}_a\Gamma_{bce}\psi_{de}+p_8\bar{\psi}_d\Gamma_{abe}\psi_{ce}+p_9\bar{\psi}_e\Gamma_{abcef}\psi_{df}\Big\}$$

$$+ih\ \mathrm{e}^{-1}\epsilon^{\alpha_1\ldots\alpha_6\mu\nu\lambda\rho}A_{\alpha_1\ldots\alpha_6}R_{\mu\nu}{}^{ab}R_{\lambda\rho}{}^{ab}$$

$$+R^{ab}\Big\{t_1H_{acd}\bar{\psi}_b\Gamma_c\psi_d+t_2H_{acd}\bar{\psi}_c\Gamma_b\psi_d+t_3H_{acd}\bar{\psi}_e\Gamma_{bcdef}\psi_f$$

$$+t_4H_{cde}\bar{\psi}_a\Gamma_{bcdef}\psi_f\Big\}$$

$$+R^{abcd}\Big\{t_5H_{abe}\bar{\psi}_e\Gamma_c\psi_d+t_6H_{abe}\bar{\psi}_c\Gamma_e\psi_d$$

$$+t_7H_{abe}\bar{\psi}_f\Gamma_{cdefg}\psi_g+t_8H_{aef}\bar{\psi}_b\Gamma_{cdefg}\psi_g+t_9H_{efg}\bar{\psi}_a\Gamma_{cdefg}\psi_b\Big\}$$

$$+H^{abc}\Big\{d_1\bar{\psi}_{ab}\Gamma_d\psi_{cd}+d_2\bar{\psi}_{ad}\Gamma_b\psi_{cd}+d_3\bar{\psi}_{de}\Gamma_{abc}\psi_{de}$$

$$+d_4\bar{\psi}_{ad}\Gamma_{bce}\psi_{de}+d_5\bar{\psi}_{ad}\Gamma_{bde}\psi_{ce}+d_6\bar{\psi}_{de}\Gamma_{abcdf}\psi_{ef}\Big\}$$

$$+e_1R^{abcd}H_{abe}H_{cde}+e_2R^{ab}H_{acd}H_{bcd}$$

$$+f_1H_{dae}H_{bce}\mathscr{D}^dH^{abc}$$

$$+g_1H_{abc}H^{abc}H_{def}H^{def}+g_2H_{acd}H_b{}^{cd}H^{aef}H^b{}_{ef}+g_3H_{abe}H_{cd}{}^eH^{acf}H^{bd}{}_f$$

$$+H^{mnp}H_{mnp}q_1\bar{\psi}_a\Gamma_b\psi_{ab}$$

$$+H^{amn}H^b{}_{mn}\Big\{q_2\bar{\psi}_c\Gamma_a\psi_{bc}+q_3\bar{\psi}_a\Gamma_c\psi_{bc}+q_4\bar{\psi}_c\Gamma_{cad}\psi_{bd}\Big\}$$

$$+H^{abm}H^{cd}{}_m\Big\{q_5\bar{\psi}_c\Gamma_d\psi_{ab}+q_6\bar{\psi}_c\Gamma_a\psi_{bd}$$

$$+q_7\bar{\psi}_e\Gamma_{abe}\psi_{cd}+q_8\bar{\psi}_e\Gamma_{ace}\psi_{bd}+q_9\bar{\psi}_a\Gamma_{cde}\psi_{be}$$

$$+q_{10}\bar{\psi}_a\Gamma_{bce}\psi_{de}+q_{11}\bar{\psi}_e\Gamma_{abc}\psi_{de}$$

$$+q_{12}\bar{\psi}_e\Gamma_{abcef}\psi_{df}+q_{13}\bar{\psi}_e\Gamma_{abcdf}\psi_{ef}\Big\}$$

$$+H^{abc}H^{def}\Big\{q_{14}\bar{\psi}_a\Gamma_{bcd}\psi_{ef}+q_{15}\bar{\psi}_a\Gamma_{bde}\psi_{cf}+q_{16}\bar{\psi}_a\Gamma_{def}\psi_{bc}$$

$$
\begin{aligned}
&+q_{17}\bar{\psi}_m\Gamma_{abcdm}\psi_{ef}+q_{18}\bar{\psi}_m\Gamma_{abcde}\psi_{fm}+q_{19}\bar{\psi}_a\Gamma_{bcdem}\psi_{fm}\\
&+q_{20}\bar{\psi}_a\Gamma_{bdefm}\psi_{cm}+q_{21}\bar{\psi}_m\Gamma_{abcdemn}\psi_{fn}\Big\}\\
&+H^{bcd}\mathscr{D}^aH_{bcd}u_1\bar{\psi}_a\Gamma_f\psi_f\\
&+H^{bcd}\mathscr{D}_bH_{cd}{}^au_2\bar{\psi}_a\Gamma_f\psi_f\\
&+H^{ade}\mathscr{D}_dH^{bc}{}_e\{u_3\bar{\psi}_a\Gamma_b\psi_c+u_4\bar{\psi}_b\Gamma_a\psi_c\}\\
&+H^{ade}\mathscr{D}^bH^c{}_{de}\Big\{u_5\bar{\psi}_a\Gamma_b\psi_c+u_6\bar{\psi}_b\Gamma_a\psi_c+u_7\bar{\psi}_a\Gamma_c\psi_b\Big\}\\
&+H^{ade}\mathscr{D}_dH^{bc}{}_eu_8\bar{\psi}_f\Gamma_{abcfg}\psi_g\\
&+H^{ade}\mathscr{D}^bH^c{}_{de}u_9\bar{\psi}_f\Gamma_{abcfg}\psi_g\\
&+H^{abf}\mathscr{D}_fH^{cde}\Big\{u_{10}\bar{\psi}_a\Gamma_{bcdeg}\psi_g+u_{11}\bar{\psi}_c\Gamma_{abdeg}\psi_g\Big\}\\
&+H^{abf}\mathscr{D}^cH^{de}{}_f\Big\{u_{12}\bar{\psi}_a\Gamma_{bcdeg}\psi_g+u_{13}\bar{\psi}_c\Gamma_{abdeg}\psi_g+u_{14}\bar{\psi}_d\Gamma_{abceg}\psi_g\Big\}\\
&+H^{abc}\mathscr{D}^dH^{efg}\Big\{u_{15}\bar{\psi}_f\Gamma_{abcde}\psi_g+u_{16}\bar{\psi}_d\Gamma_{abcef}\psi_g+u_{17}\bar{\psi}_a\Gamma_{bcdef}\psi_g\\
&+u_{18}\bar{\psi}_a\Gamma_{bdefg}\psi_c+u_{19}\bar{\psi}_d\Gamma_{abefg}\psi_c+u_{20}\bar{\psi}_i\Gamma_{abcdefgij}\psi_j\Big\}\\
&+\bar{\psi}^a\Gamma^b\psi^c\{w_1H_{abc}H_{mnp}H_{mnp}+w_2H_{abm}H_{cnp}H_{mnp}\\
&+w_3H_{acm}H_{bnp}H_{mnp}+w_4H_{amn}H_{bnp}H_{cpm}\}\\
&+\bar{\psi}_e\Gamma^{abcef}\psi_f\{w_5H_{abc}H_{mnp}H_{mnp}+w_6H_{abm}H_{cnp}H_{mnp}\\
&+w_7H_{amn}H_{bnp}H_{cpm}\}\\
&+\bar{\psi}^e\Gamma^{abcdf}\psi_f\{w_8H_{abc}H_{dmn}H_{emn}+w_9H_{abm}H_{cen}H_{dmn}\}\\
&+\bar{\psi}^f\Gamma^{abcde}\psi^g\{w_{10}H_{abc}H_{dem}H_{fgm}+w_{11}H_{abc}H_{dfm}H_{egm}\\
&+w_{12}H_{abf}H_{cdm}H_{egm}+w_{13}H_{afg}H_{bcm}H_{dem}\}\\
&+\bar{\psi}_i\Gamma^{abcdefgij}\psi_jw_{14}H_{abc}H_{dem}H_{fgm}.
\end{aligned}
\tag{B.1}
$$

In the terms bilinear in the gravitino (those with coefficients t, u and w) no terms with $\Gamma^{(3)}$ or $\Gamma^{(7)}$ appear. The reason is, that these are *symmetric* in the two gravitinos, and therefore give upon partial integration $\mathscr{D}_\mu\psi_\nu+\mathscr{D}_\nu\psi_\mu$ in the variation of the action. Since there are no other sources for variations, these terms all have to vanish.

In (B.1) the coefficients u_4, u_{14}, u_{15} and u_{18} can be set equal to zero because by partial integration the corresponding terms can be rexpressed in terms of other contributions to the ansatz. A similar mechanism for the p-contributions was discussed in sect. 3.

We now rewrite the above ansatz, with the values we obtained for the coefficients. The arbitrary scale (a_0) has been set equal to one. This next formula is the main result of this paper:

$$
\begin{aligned}
\mathrm{e}^{-1}\mathscr{L} = {} & R_{\mu\nu}{}^{ab}R^{\mu\nu ab} - 2R_{\mu\nu}R^{\mu\nu} + \bar{\psi}^{ab}\not{\mathscr{D}}\psi_{ab} - 3\mathscr{D}_\mu H_{abc}\mathscr{D}^\mu H^{abc} \\
& + R^{abcd}\left\{-\bar{\psi}_a\Gamma_b\psi_{cd} + \tfrac{3}{4}\bar{\psi}_e\Gamma_{abe}\psi_{cd} + \tfrac{1}{2}\bar{\psi}_a\Gamma_{cde}\psi_{be}\right\} \\
& + R^{ab}\left\{+\bar{\psi}_a\Gamma_c\psi_{bc} - \bar{\psi}_c\Gamma_{cad}\psi_{bd}\right\} \\
& + \sqrt{2}\mathscr{D}^d H^{abc}\left\{\tfrac{3}{2}\bar{\psi}_d\Gamma_a\psi_{bc} - \tfrac{1}{6}\bar{\psi}_e\Gamma_{abc}\psi_{de} + \tfrac{3}{2}\bar{\psi}_a\Gamma_{bce}\psi_{de} - \tfrac{3}{2}\bar{\psi}_d\Gamma_{abe}\psi_{ce}\right\} \\
& + i\sqrt{2}\,\mathrm{e}^{-1}\,\epsilon^{\alpha_1\ldots\alpha_6\mu\nu\lambda\rho}A_{\alpha_1\ldots\alpha_6}R_{\mu\nu}{}^{ab}R_{\lambda\rho}{}^{ab} \\
& + \sqrt{2}\,R^{ab}\left\{\tfrac{3}{2}H_{acd}\bar{\psi}_b\Gamma_c\psi_d - \tfrac{9}{4}H_{acd}\bar{\psi}_c\Gamma_b\psi_d + \tfrac{3}{8}H_{acd}\bar{\psi}_e\Gamma_{bcdef}\psi_f\right. \\
& \left. - \tfrac{1}{4}H_{cde}\bar{\psi}_a\Gamma_{bcdef}\psi_f\right\} \\
& + \sqrt{2}\,R^{abcd}\left\{-\tfrac{9}{4}H_{abe}\bar{\psi}_e\Gamma_c\psi_d - \tfrac{1}{8}H_{abe}\bar{\psi}_c\Gamma_e\psi_d\right. \\
& \left. - \tfrac{3}{8}H_{aef}\bar{\psi}_b\Gamma_{cdefg}\psi_g - \tfrac{5}{48}H_{efg}\bar{\psi}_a\Gamma_{cdefg}\psi_b\right\} \\
& + \sqrt{2}\,H^{abc}\left\{6\bar{\psi}_{ab}\Gamma_d\psi_{cd} - \tfrac{3}{2}\bar{\psi}_{ad}\Gamma_b\psi_{cd} - \tfrac{1}{12}\bar{\psi}_{de}\Gamma_{abc}\psi_{de}\right. \\
& \left. - 3\bar{\psi}_{ad}\Gamma_{bde}\psi_{ce} - \tfrac{1}{4}\bar{\psi}_{de}\Gamma_{abcdf}\psi_{ef}\right\} \\
& + 9R^{ab}H_{acd}H_{bcd} \\
& + 27\sqrt{2}\,H_{dae}H_{bce}\mathscr{D}^d H^{abc} \\
& + \tfrac{27}{2}H_{abe}H_{cd}{}^{e}H^{acf}H^{bd}{}_{f} \\
& + H^{mnp}H_{mnp}\tfrac{5}{12}\bar{\psi}_a\Gamma_b\psi_{ab} \\
& + H^{amn}H^{b}{}_{mn}\left\{\tfrac{1}{2}\bar{\psi}_c\Gamma_a\psi_{bc} - \tfrac{7}{2}\bar{\psi}_a\Gamma_c\psi_{bc} + \tfrac{7}{2}\bar{\psi}_c\Gamma_{cad}\psi_{bd}\right\} \\
& + H^{abm}H^{cd}{}_{m}\left\{-7\bar{\psi}_c\Gamma_d\psi_{ab} - 11\bar{\psi}_c\Gamma_a\psi_{bd}\right. \\
& + \tfrac{9}{4}\bar{\psi}_e\Gamma_{abe}\psi_{cd} - \tfrac{19}{4}\bar{\psi}_e\Gamma_{ace}\psi_{bd} + \tfrac{21}{4}\bar{\psi}_a\Gamma_{cde}\psi_{be}
\end{aligned}
$$

$$
\begin{aligned}
&+14\bar{\psi}_a\Gamma_{bce}\psi_{de}+\tfrac{1}{2}\bar{\psi}_e\Gamma_{abc}\psi_{de}\\
&+\tfrac{9}{4}\bar{\psi}_e\Gamma_{abcef}\psi_{df}-\tfrac{5}{8}\bar{\psi}_e\Gamma_{abcdf}\psi_{ef}\Big\}\\
&+H^{abc}H^{def}\Big\{+5\bar{\psi}_a\Gamma_{bde}\psi_{cf}+\tfrac{3}{4}\bar{\psi}_a\Gamma_{def}\psi_{bc}\\
&-\tfrac{13}{24}\bar{\psi}_m\Gamma_{abcdm}\psi_{ef}-\tfrac{1}{12}\bar{\psi}_m\Gamma_{abcde}\psi_{fm}-\tfrac{1}{2}\bar{\psi}_a\Gamma_{bcdem}\psi_{fm}\\
&+\tfrac{1}{4}\bar{\psi}_a\Gamma_{bdefm}\psi_{cm}-\tfrac{1}{6}\bar{\psi}_m\Gamma_{abcdemn}\psi_{fn}\Big\}\\
&-H^{bcd}\mathscr{D}_bH_{cd}{}^a\bar{\psi}_a\Gamma_f\psi_f\\
&-4H^{ade}\mathscr{D}_dH^{bc}{}_e\bar{\psi}_a\Gamma_b\psi_c\\
&-H^{ade}\mathscr{D}^bH^c{}_{de}\bar{\psi}_a\Gamma_b\psi_c\\
&-\tfrac{1}{2}H^{ade}\mathscr{D}_dH^{bc}{}_e\bar{\psi}_f\Gamma_{abcfg}\psi_g\\
&+\tfrac{1}{8}H^{ade}\mathscr{D}^bH^c{}_{de}\bar{\psi}_f\Gamma_{abcfg}\psi_g\\
&+H^{abf}\mathscr{D}_fH^{cde}\Big\{\tfrac{1}{3}\bar{\psi}_a\Gamma_{bcdeg}\psi_g+\tfrac{1}{2}\bar{\psi}_c\Gamma_{abdeg}\psi_g\Big\}\\
&-H^{abf}\mathscr{D}^cH^{de}{}_f\bar{\psi}_a\Gamma_{bcdeg}\psi_g\\
&+H^{abc}\mathscr{D}^dH^{efg}\Big\{+\tfrac{1}{2}\bar{\psi}_a\Gamma_{bcdef}\psi_g\\
&+\tfrac{1}{6}\bar{\psi}_a\Gamma_{bdefg}\psi_c-\tfrac{1}{144}\bar{\psi}_i\Gamma_{abcdefgij}\psi_j\Big\}\\
&+\sqrt{2}\,\bar{\psi}^a\Gamma^b\psi^c\Big\{-\tfrac{11}{16}H_{abc}H_{mnp}H_{mnp}+\tfrac{45}{4}H_{abm}H_{cnp}H_{mnp}\\
&+3H_{acm}H_{bnp}H_{mnp}-\tfrac{39}{8}H_{amn}H_{bnp}H_{cpm}\Big\}\\
&+\sqrt{2}\,\bar{\psi}_e\Gamma^{abcef}\psi_f\Big\{\tfrac{19}{288}H_{abc}H_{mnp}H_{mnp}-\tfrac{9}{16}H_{abm}H_{cnp}H_{mnp}\Big\}\\
&+\sqrt{2}\,\bar{\psi}_e\Gamma^{abcdf}\psi_f\Big\{-\tfrac{17}{24}H_{abc}H_{dmn}H_{emn}-\tfrac{7}{8}H_{abm}H_{cen}H_{dmn}\Big\}\\
&+\sqrt{2}\,\bar{\psi}^f\Gamma^{abcde}\psi^g\Big\{\tfrac{1}{16}H_{abc}H_{dem}H_{fgm}-\tfrac{7}{48}H_{abc}H_{dfm}H_{egm}\\
&+\tfrac{19}{8}H_{abf}H_{cdm}H_{egm}-\tfrac{19}{32}H_{afg}H_{bcm}H_{dem}\Big\}\\
&-\tfrac{1}{64}\sqrt{2}\,\bar{\psi}_i\Gamma^{abcdefgij}\psi_jH_{abc}H_{dem}H_{fgm}.
\end{aligned}
\tag{B.2}
$$

For convenience of the reader the terms in (B.2) are ordered in the same way as in the ansatz (B.1). Undoubtedly the result can be simplified somewhat by absorbing some of the terms with explicit gravitinos into supercovariantizations, or by combing terms by using identities for the Γ-matrices. Since we have no systematic way of proceeding with such efforts, we prefer to present the result in the form it was obtained.

References

[1] E.S. Fradkin and A.A. Tseytlin, Phys. Rep. 119 (1985) 233
[2] M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D17 (1978) 3179;
P.K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D19 (1979) 3166
[3] E. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. B182 (1981) 173
[4] E. Bergshoeff, A. Salam and E. Sezgin, Nucl. Phys. B279 (1987) 659
[5] E. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. B217 (1983) 489
[6] A.H. Chamseddine, Nucl. Phys. B185 (1981) 403;
E. Bergshoeff, M. de Roo, B. de Wit and P. van Niewenhuizen, Nucl. Phys. B195 (1982) 97;
G.F. Chapline and N.S. Manton, Phys. Lett. B120 (1983) 105
[7] M.B. Green and J.H. Schwarz, Phys. Lett. B149 (1984) 117
[8] A. Salam and E. Sezgin, ed., Supergravities in diverse dimensions (North-Holland/World Scientific, Amsterdam/Singapore, 1989)
[9] E. Bergshoeff and M. Rakowski, Phys. Lett. B191 (1987) 399
[10] E. Bergshoeff and M. de Roo, Phys. Lett. B218 (1989) 210;
E.A. Bergshoeff and M. de Roo, Nucl. Phys. B328 (1989) 439
[11] A.H. Chamseddine, Phys. Rev. D24 (1981) 3065
[12] S.J. Gates Jr. and H. Nishino, Phys. Lett. B173 (1986) 46
[13] A.H. Chamseddine and Pran Nath, Phys. Rev. D34 (1986) 3769;
S.J. Gates Jr. and H. Nishino, Phys. Lett. B173 (1986) 52;
R. D'Auria and P. Fré, Mod. Phys. Lett. A3 (1988) 673;
E. Bergshoeff and M. de Roo, Phys. Lett. B247 (1990) 530
[14] B. de Wit, in Supersymmetry, supergravity and superstrings '86, ed. B. de Wit, P. Fayet and M.T. Grisaru (World Scientific, Singapore, 1986)
[15] P. Howe, H. Nicolai and A. Van Proeyen, Phys. Lett. B112 (1982) 446
[16] S.J. Gates Jr. and H. Nishino, Phys. Lett B266 (1991) 14
[17] E. Bergshoeff and M. de Roo, Phys. Lett. B112 (1982) 53
[18] E. Bergshoeff, E. Sezgin and A. Van Proeyen, Nucl. Phys. B264 (1986) 653
[19] B. Zwiebach, Phys. Lett. B156 (1985) 315